Lecture 6: Subspaces

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5 Subspaces

5.1 Definition and test

Let V be a vector space.
Let W is a subset of V: $W \subseteq V$

Def^n: W is called a subspace of V, written $W \subseteq V$, if W equipped with addition and scalar multiplication of V, is itself a vector space.

Note
$\{(x, 2x) | \ x \ \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$ (with standard operations)

Theorem (subspace test)
Let V be a vector space. A subset $W \subseteq$ V is a subspace of V if and only if the following hold:

(i) $0 \epsilon W$
(ii) $\forall \ u, v \in W: u + v \ \in W$ ("closed under addition")
(iii) $\forall \ \ c \ \in \mathbb{R}, \forall \ \ u \in W: \ \ c * u \ \in$ W ("closed under scalar multiplication")

5.2 Examples

a) planes through (0,0,0) in $\mathbb{R}^3$ are subspaces.
$T = \{u \in \mathbb{R}^3 | \ u * n = 0\}$ fixed normal vector $n \neq 0$

Apply the subspace test:

(i) Since $(0,0,0) \ * n = 0$, we know that (0,0,0) $\in T$
(ii) Let $u, v \ \in T$. Then $(u + v) \cdot n = u \cdot n + v \cdot n = 0$. So $u, v \ \in T$
(iii) Let $c \in \mathbb{R}, u \in T$. Then, $(c * u) \cdot n = c * 0 = 0$. So, $c * u \ \in T$

Note In $\mathbb{R}^3$: any plane through $(0,0,0)$ $\rightarrow$ is a subspace
any plane not through $(0,0,0)$ $\rightarrow$ isn't a subspace

Now, in $\mathbb{R}^n$, any line through $(0, \ldots, 0)$ $\rightarrow$ subspace

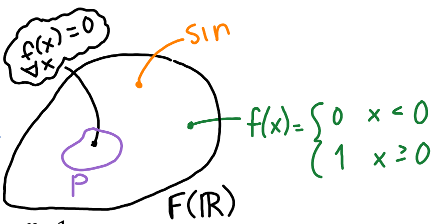
PROOF: Let $d \in \mathbb{R}, d \neq 0$, be a direction vector.

$L = \{t * d \mid t \in \mathbb{R}\}$

i. $(0, \ldots, 0) = 0 * d \quad \in L$
ii. $t_1 * d + t_2 * d = (t_1 + t_2) * d \quad \in L$
iii. $c * (td) = (ct)d \quad \in L$

$\therefore$ L is a subspace of $\mathbb{R}^n$.

Another example:
$V = \mathrm{F}(\mathbb{R})$ with "+" and "·" as before
any function is a point in the vector space



$\mathrm{P} = \{\text{polynomial functions } \mathbb{R} \rightarrow \mathbb{R}\} \leq \mathrm{F}(\mathbb{R})$

P is subspace $\mathrm{F}(\mathbb{R})$ $\quad p(x) = a_n x^n, a_{n-1} x^{n-1}, \ldots, a_1 x + a_0$ with $a \in \mathbb{R}$

Another example:
$V = \mathrm{M}_{22}(\mathbb{R})$ $\rightarrow$ matrices 2x2 size whose entries are real
Def^n: The transpose of a m*n matrix A is the n*m matrix A^T whose rows are the columns of A.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Now consider the set:
$S \coloneqq \{\mathrm{A} \ \in \mathrm{M}_{22}(\mathbb{R}) | \ \mathrm{A}^{\mathrm{T}} = \mathrm{A}\} \ \subseteq \mathrm{M}_{22}(\mathbb{R})$

Now consider the set:

$S \coloneqq \{\mathrm{A} \in \mathrm{M}_{22}(\mathbb{R}) \mid \mathrm{A}^{\mathrm{T}} = \mathrm{A}\} \subseteq \mathrm{M}_{22}(\mathbb{R})$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$\mathrm{A}^T = \mathrm{A}$ means $\mathrm{b} = \mathrm{c}$ ie. $\begin{bmatrix} 1 & 7 \\ 7 & 2 \end{bmatrix}$

In this case, we call A symmetric.

Subspace test:

(i) $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, therefore (i) is satisfied

result is a matrix w/ same entries in upper right and lower left

(ii) $\mathrm{A} = \begin{bmatrix} a_1 & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$ $\mathrm{B} = \begin{bmatrix} b_1 & b_{12} \\ b_{12} & b_{22} \end{bmatrix}$ $\mathrm{A} + \mathrm{B} = \begin{bmatrix} * & a_{12} b_{12} \\ a_{12} b_{12} & * \end{bmatrix}$

therefore (ii) is satisifed

(iii) same idea as (ii), scalar multiplication is satisfied